CSC 412/2506:
Probabilistic Learning and Reasoning
Week 3 - 1/2: Markov Random Fields

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Overview

- Markov Random Fields (MRFs)
- Assignment 1 is released today.
- TA office hours next week.

Are DAGMs always useful?

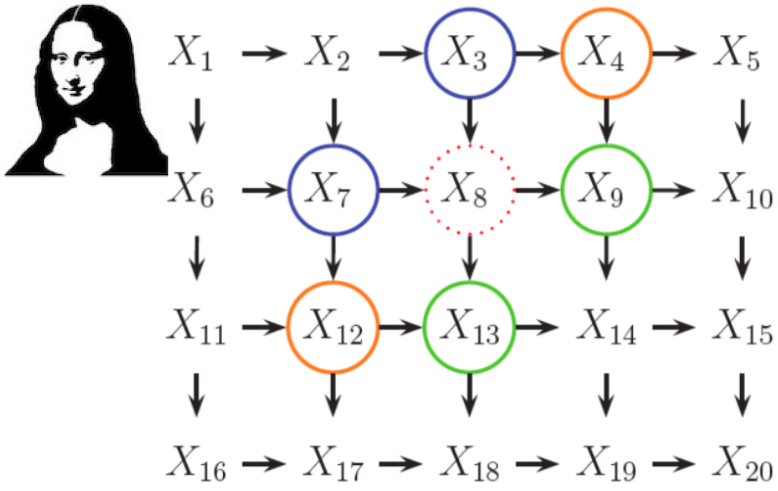


Figure : Causal MRF or a Markov mesh

- Each node is conditionally independent of its ancestors (non-descendents that are not parents) given its parents

$$\{x_i \perp \text{ancestors}(x_i) \mid \text{parents}(x_i)\} \quad \forall i.$$

- For some problems, it is not clear how to choose the edge directions in DAGMs.

$$mb(X_8) = \{X_3, X_7\} \cup \{X_9, X_{13}\} \cup \{X_{12}, X_4\}$$

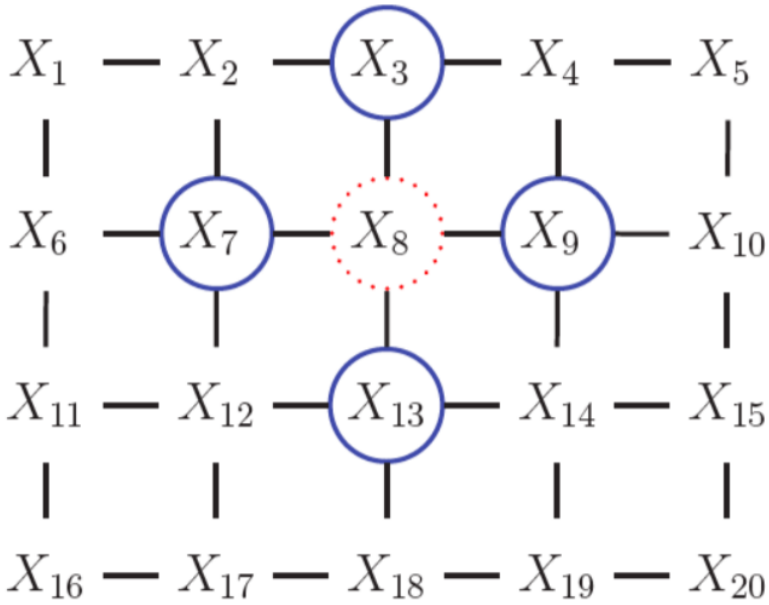
Markov blanket

The set of nodes that renders a node conditionally independent of all the other nodes in the graph is called that node's Markov blanket (mb).

Above, one would expect X_4 and X_{12} not to be in the Markov blanket $mb(X_8)$, especially given X_2 and X_{14} are not.

Markov Random Fields

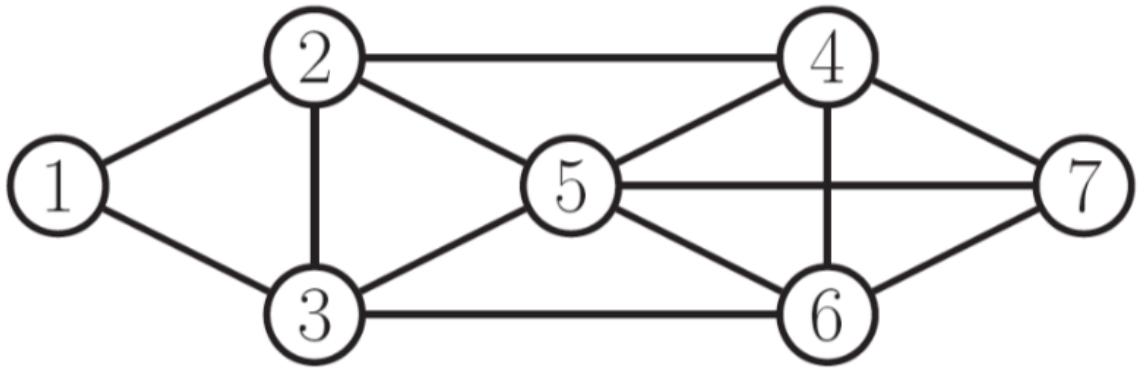
- Undirected graphical models, also called Markov random fields (MRFs), are a set of random variables where the dependencies are described by an undirected graph.
- The nodes in the graph represent random variables. However, in contrast to DAGMs, edges represent probabilistic interactions between neighbors (as opposed to conditional dependence).



Dependencies in MRFs

The following 3 properties determine if nodes are conditionally independent in MRFs:

1. **Global Markov Property** (G): $x_A \perp x_B | x_C$ iff x_C separates x_A from x_B. That is, there is no path in the graph between A and B that doesn't go through x_C.



$$\{x_1, x_2\} \perp \{x_6, x_7\} | \{x_3, x_4, x_5\}$$

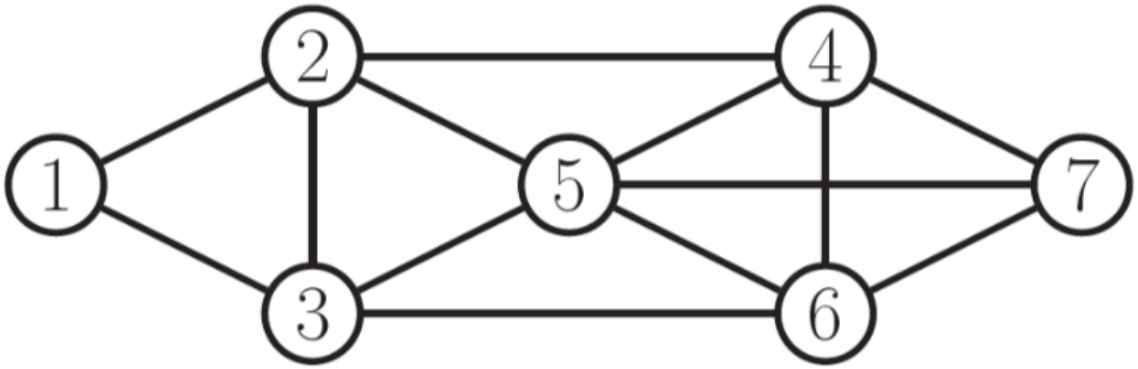
Dependencies in MRFs

The following 3 properties determine if nodes are conditionally independent in MRFs:

2 **Local Markov Property (Markov Blanket)** (L): The set of nodes that renders a node t conditionally independent of all the other nodes in the graph

$$t \perp (\text{all } \setminus cl(t)) \mid mb(t)$$

where $cl(t) = mb(t) \cup t$ is the closure of node t.



$$x_1 \perp \text{rest} | \{x_2, x_3\}$$

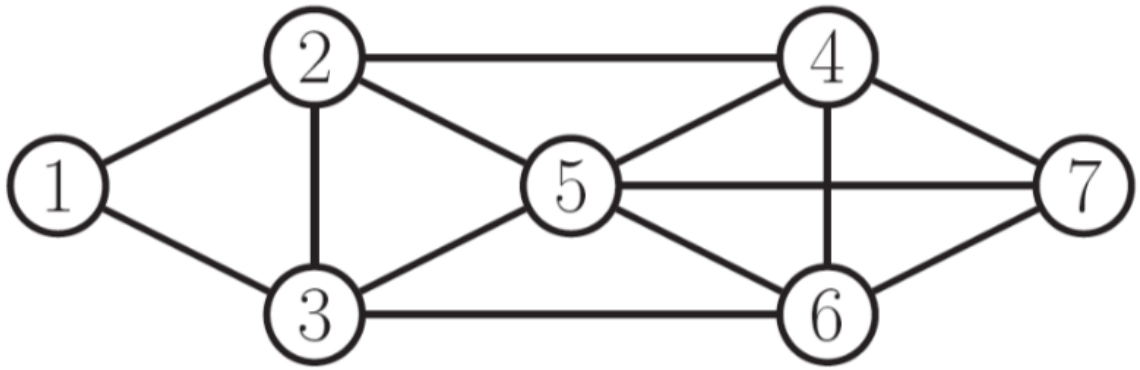
so $mb(x_1) = \{x_2, x_3\}$.

Dependencies in MRFs

The following 3 properties determine if nodes are conditionally independent in MRFs:

3 **Pairwise (Markov) Property** (P): The set of nodes that renders two nodes, s and t, conditionally independent of each other.

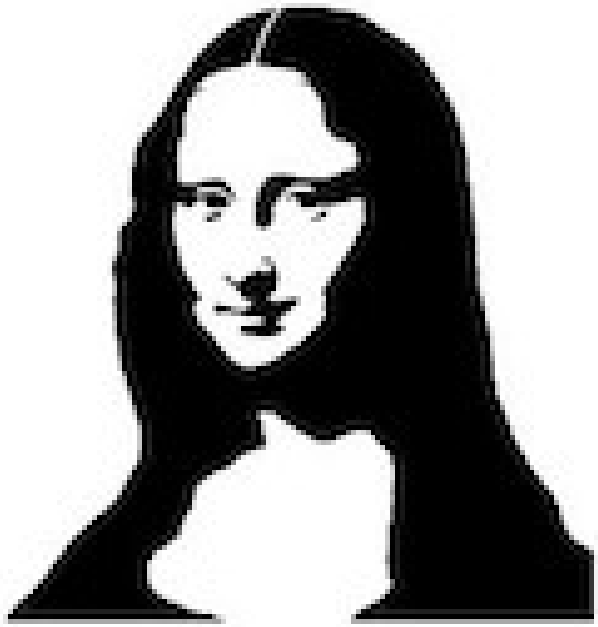
$$s \perp t | (\text{all} \setminus \{s, t\}) \Leftrightarrow \text{No edge between } s \& t.$$

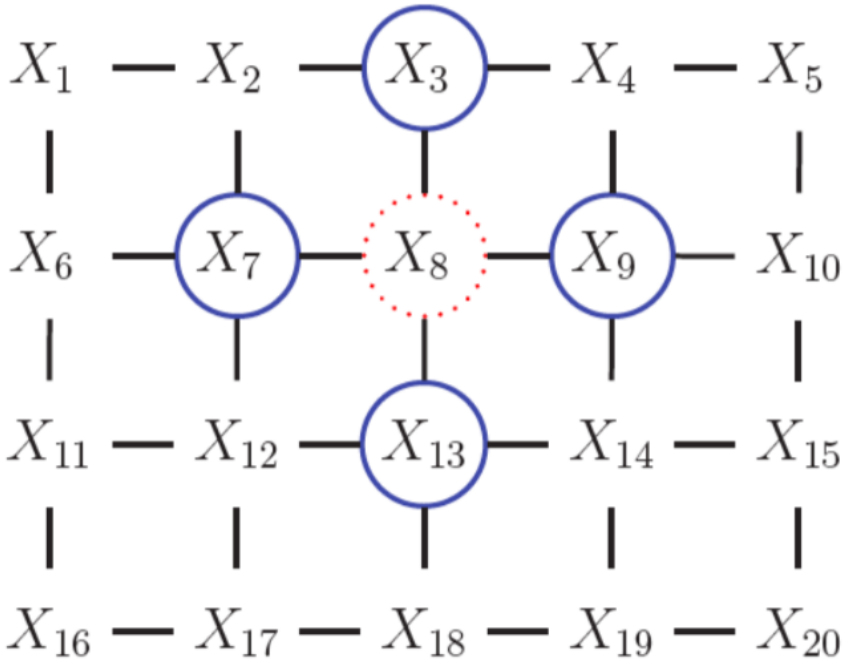


$$x_1 \perp x_7 | \text{rest}$$

iff there are no edges between x_1 and x_7. E.g. $1 \not\perp 2 | \text{rest}$

Image MRF

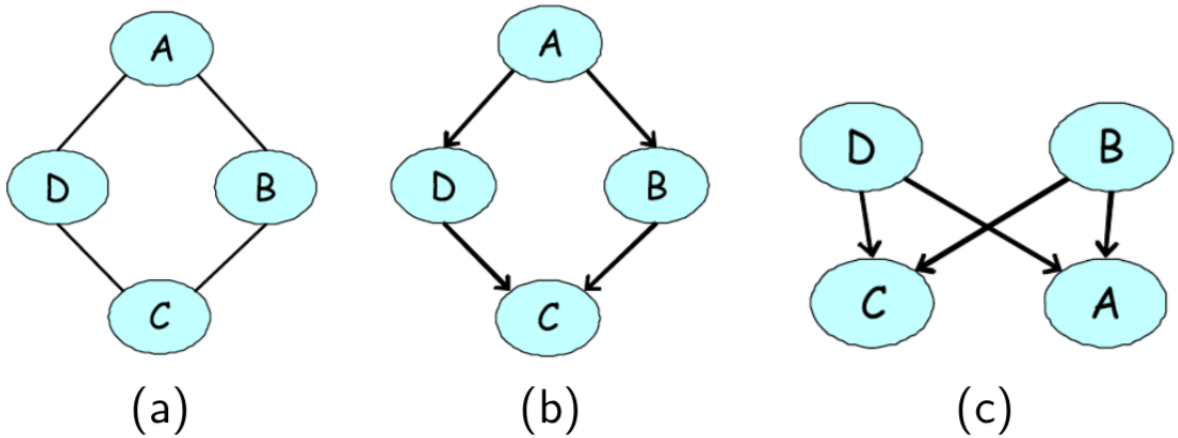




- Global: $\{X_1, X_2\} \perp \{X_{15}, X_{20}\} | \{X_3, X_6, X_7\}$
- Local: $X_1 \perp \text{rest} | \{X_2, X_6\}$
- Pairwise: $X_1 \perp X_{20} | \text{rest}$

Not all MRFs can be represented as DAGMs

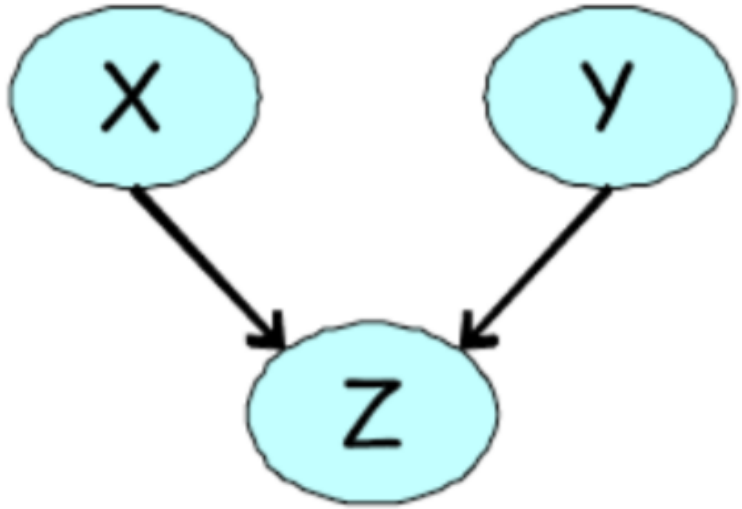
Take the following MRF for example (a) and our attempts at encoding this as a DAGM (b, c).



- Two conditional independencies in (a):
  - 1. $A \perp C | D, B$ 2. $B \perp D | A, C$
- In (b), we have the first independence, but not the second.
- In (c), we have the first independency, but not the second. Also, B and D are marginally independent.

Not all DAGMs can be represented as MRFs

Not all DAGMs can be represented as MRFs.
E.g. explaining away:

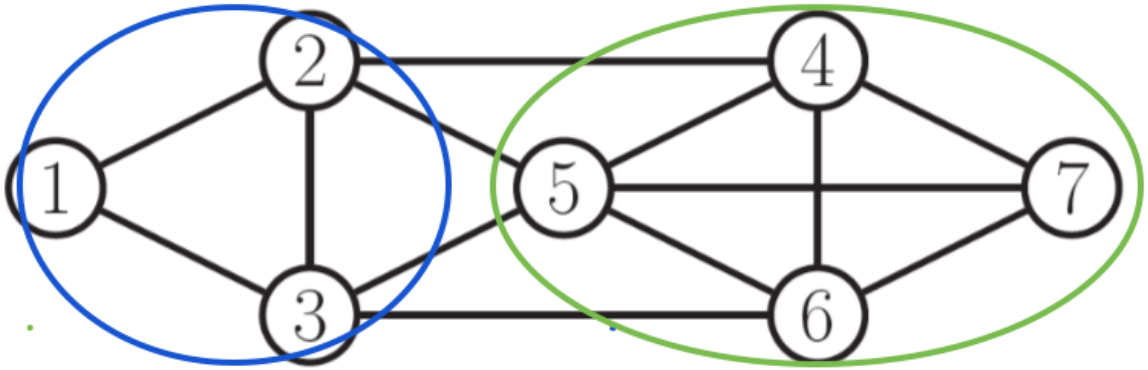


An undirected model is unable to capture the marginal independence, $X \perp Y$ that holds at the same time as $X \not\perp Y | Z$.

Cliques

A **clique** is a subset of notes such that every two vertices in the subset are connected by an edge.

- **Maximal clique** is a clique that cannot be extended by including one more adjacent vertex.
- **Maximum clique** is a clique of the largest possible size in a given graph.



Above, maximal clique is show in blue, while a maximum clique is shown in green.

Distributions Induced by MRFs

Let $x = (x_1, ..., x_m)$ be the set of all random variables in our graph.

- Unlike in DAGMs, there is no topological ordering in MRFs, so the chain rule cannot be used to simplify the joint dist $p(x)$.
- We associate potential functions with each maximal clique: given a maximal clique c, the potential function (or factor)

$$\psi_c(x_c|\theta_c)$$

  is a non-negative function, where x_c is the subset of variables in c and θ_c is some parameter.
- Joint distribution is proportional to product of clique potentials

$$p(x) \propto \prod_{c \in \mathcal{C}} \psi_c(x_c|\theta_c)$$

  where $\mathcal{C}$ is the set of all maximal cliques.

Any distribution whose conditional independencies are represented with an MRF can be represented this way.

Distributions Induced by MRFs

- A distribution $p(x) > 0$ satisfies the conditional independence properties of an undirected graph iff $p(x)$ can be represented as a product of factors, one per maximal clique, i.e.,

$$p(x|\theta) = \frac{1}{Z(\theta)} \prod_{c \in \mathcal{C}} \psi_c(x_c|\theta_c)$$

  where $\mathcal{C}$ is the set of all (maximal) cliques of G, and $Z(\theta)$ the **partition function**, defined as

$$Z(\theta) = \sum_x \prod_{c \in \mathcal{C}} \psi_c(x_c|\theta_c).$$

- The factored structure of the distribution makes it possible to more efficiently do the sums/integrals needed to compute it.

MRFs as Exponential Families

- We can write this as an exponential family:

$$p(x|\theta) = \exp\left\{ \sum_{c \in \mathcal{C}} \log \psi_c(x_c|\theta_c) - \underbrace{\log Z(\theta)}_{=A(\theta)} \right\}$$

- If the potentials have a log-linear form (model assumption)

$$\log \psi_c(x_c|\theta_c) = \theta_c^T \phi_c(x_c)$$

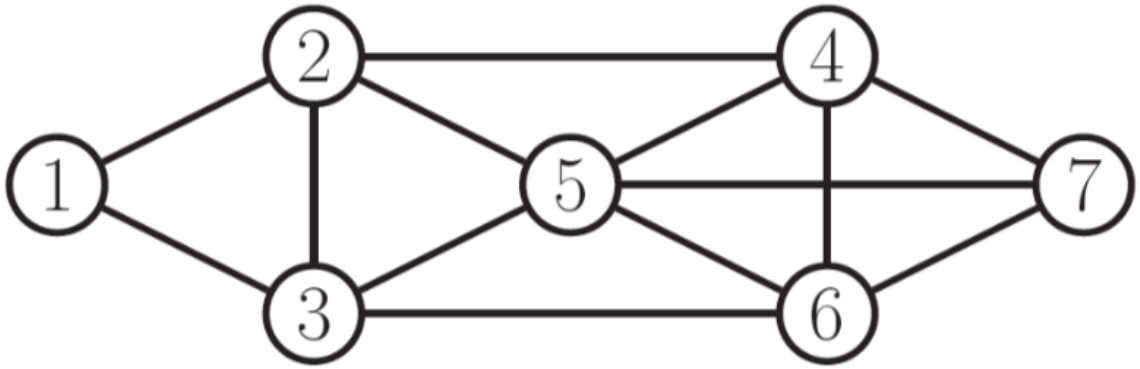
we get

$$p(x|\theta) = \exp\left\{ \sum_{c \in \mathcal{C}} \theta_c^T \phi_c(x_c) - \underbrace{\log Z(\theta)}_{=A(\theta)} \right\}$$

- Thus, we can find the expectation of the c-th feature

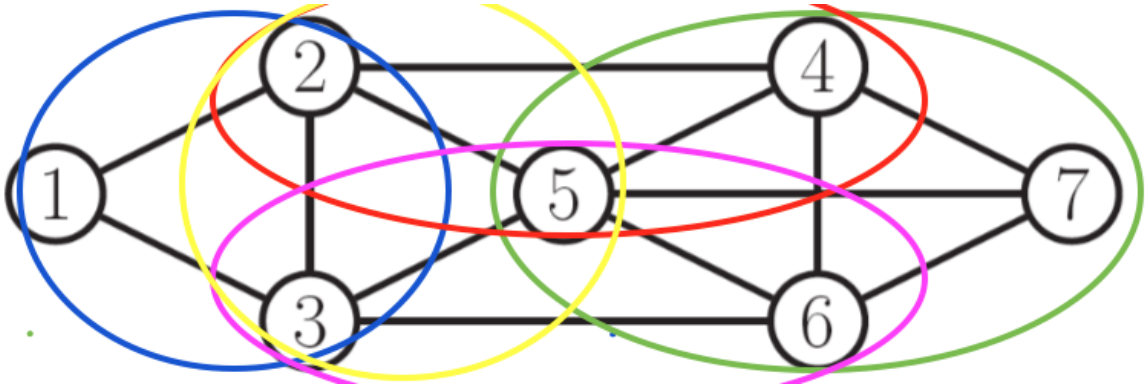
$$\frac{\partial \log Z(\theta)}{\partial \theta_c} = \mathbb{E}[\phi_c(x_c)]$$

Example:



- How to factorize the undirected graph of our running example?
- How many maximal cliques are there ?

Example:



Lets see how to factorize the undirected graph of our running example:

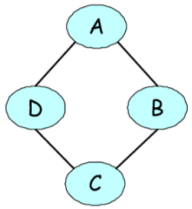
$$p(x) \propto \psi_{1,2,3}(x_1, x_2, x_3)\psi_{2,3,5}(x_2, x_3, x_5)\psi_{2,4,5}(x_2, x_4, x_5)\psi_{3,5,6}(x_3, x_5, x_6) \times \psi_{4,5,6,7}(x_4, x_5, x_6, x_7)$$

Representing potentials

If the variables are discrete, we can represent the potential functions as tables of (non-negative) numbers

$$p(A, B, C, D) = \frac{1}{Z}\psi_{A,B}(A, B)\psi_{B,C}(B, C)\psi_{C,D}(C, D)\psi_{A,D}(A, D)$$

where



$\psi_{AB}[A, B]$

a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10

$\psi_{BC}[B, C]$

b^0	c^0	100
b^0	c^1	1
b^1	c^0	1
b^1	c^1	100

$\psi_{CD}[C, D]$

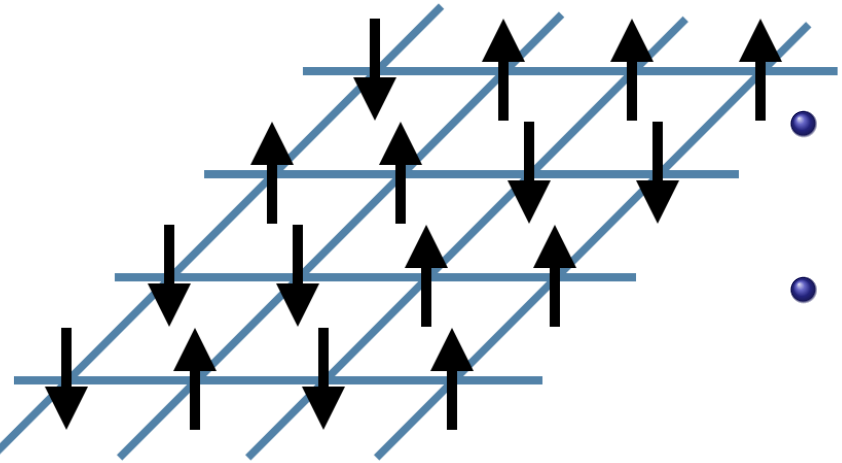
c^0	d^0	1
c^0	d^1	100
c^1	d^0	100
c^1	d^1	1

$\psi_{AD}[D, A]$

d^0	a^0	100
d^0	a^1	1
d^1	a^0	1
d^1	a^1	100

Note that these potentials are not probabilities, but instead encode relative affinities between the different assignments. For example, in the above table, a^0, b^0 is taken to be 30X more likely than a^1, b^0.

Example: Ising model



- The Ising model is an MRF that is used to model magnets.
- The nodes variables are spins, i.e., we use $x_s \in \{-1, +1\}$.

- Define the pairwise clique potentials as

$$\psi_{st}(x_s, x_t) = \begin{pmatrix} e^{W_{st}} & e^{-W_{st}} \\ e^{-W_{st}} & e^{W_{st}} \end{pmatrix}$$

  where W_{st} is the coupling strength between nodes s and t.
- For $(x_s, x_t) = (1, -1)$ we have $\psi_{st}(1, -1) = e^{-W_{st}}$ or in compact form $\psi_{st}(x_s, x_t) = e^{x_s x_t W_{st}}$.
- If two nodes are connected we set $W_{st} = J$ and if they are not connected $W_{st} = 0$.
- This captures: it is more likely to have the same neighboring spins.

Ising model

- In compact form, for all pairs (s, t), we can write

$$\psi_{st}(x_s, x_t) = e^{x_s x_t W_{st}} = \text{pairwise potential}$$

- This only encodes the pairwise behavior. We might want to add node potentials as well

$$\psi_s(x_s) = e^{b_s x_s}$$

- The overall distribution becomes

$$p(x) \propto \prod_{s \sim t} \psi_{st}(x_s, x_s) \prod_s \psi_s(x_s) = \exp\left\{ J \sum_{s \sim t} x_s x_t + \sum_s b_s x_s \right\}.$$

- If $x_s = x_t$, i.e. the neighboring spins are the same, then the likelihood is larger. We will use this Ising model for image denoising in the Assignment 2!

Summary

- In certain cases, DAGMs cannot encode the proper conditional independences.
- MRFs are useful if there is no topological ordering in the graph (image).
- Cliques are key to parametrizing distributions induced by MRFs.
- Ising model is one useful MRF example.
- Next week: how to do exact inference?